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Representations of quivers on abelian categories and monads on projective varieties

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Abstract. We consider representations of quivers in arbitrary categories and twisted representations of quivers in arbitrary tensor categories. We show that if $\mathcal{A}$ is an abelian category, then the category of representations of a quiver in $\mathcal{A}$ is also abelian, and that the category of twisted linear representations of a quiver is equivalent to the category of linear (untwisted) representations of a different quiver. We conclude by discussing how representations of quivers arise naturally in certain important problems concerning monads ans sheaves on projective varieties.

1. Introduction

Quivers are a valuable tool in the theory of finite dimensional associative algebras and their representations. Moreover, linear representations of quivers are a beautiful subject in itself, with many interesting connections with other areas. Recently, many authors have considered representations of quivers in arbitrary categories, see in particular [8], motivated by the relevance of such concept in algebraic geometry and mathematical physics. More precisely, representations of quivers in the category of vector bundles or coherent sheaves on a projective variety, called quiver bundles or quiver sheaves, unify many of the vector bundles with extra structure which have been previously considered in the literature (e.g. Higgs bundles, coherent systems, holomorphic triples, etc), see [2, 8]. Several recent papers also consider quiver bundles in connection with string theory, see for instance [4, 16].

In this paper, we consider representations of quivers in arbitrary categories and twisted representations of quivers in arbitrary tensor categories, as well as some applications of these concepts to the study of certain problems concerning vector bundles over projective varieties.

First, we prove that if $\mathcal{A}$ is an additive (abelian) category, then the category of representations of a quiver in $\mathcal{A}$ is also additive (abelian), showing that the category of representations often inherits some of the properties of the original category. We also discuss how functors between categories induce functors between the respective categories of representations, and show that such induced functors also inherit properties from the original ones.

Next, we consider twisted representations of quivers in arbitrary tensor categories, and show that the category of twisted linear representations of a quiver is equivalent to the category of linear (untwisted) representations of a different quiver.

Finally, we turn to one of the original motivations behind this project by discussing the theory of monads on a projective variety X from a categorical point of view, and showing that these can be regarded as representations of a quiver with relation in the category of vector bundles over X. We then focus on a particular class of sheaves on projective space, so-called linear sheaves, and show that the category of such sheaves is equivalent to a subcategory of the category of twisted linear representations of a quiver with relations. We look at examples in which geometric properties of sheaves are translated into algebraic properties of the corresponding twisted representations, and vice versa. This leads to a possibly (and hopefully) fruitful application of representation theoretical methods to the study of vector bundles over projective varieties.

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2. Representations of Quivers

Recall that a *quiver* Q consists of a pair (Q_0, Q_1) where Q_0 is the set of vertices and Q_1 is the set of arrows, and we have two maps $t, h : Q_1 \to Q_0$ named tail and head. A path in the quiver is a sequence of arrows

$$p = a_1 a_2 \cdots a_n$$

such that $h(a_{i+1}) = t(a_i)$, $i = 1, \cdots, n-1$.

$$\xrightarrow{a_n} \xrightarrow{a_{n-1}} \cdots \xrightarrow{a_2} \xrightarrow{a_1}$$

We say that the path starts in $t(a_n)$ and ends in $h(a_1)$. The *path algebra* kQ is the associative algebra generated by all paths of Q with the product given by concatenation of paths. A *relation* R in Q is a sum of paths $p_i \in kQ$, $R = \sum_{i=1}^{n} p_i$, such that $t(p_i) = t(p_j)$ and $h(p_i) = h(p_j)$, $i, j = 1, \ldots, n$. Given some relations $R_j = \sum_{i=1}^{n_j} p_i^j$, $j = 1, \ldots, m$, $p_i^j \in kQ$, the *path algebra with relations* is given by the quotient kQ/I, where I is the ideal of kQ generated by the relations R_j, $j = 1, \ldots, m$.

2.1. Representations of quivers in additive and abelian categories.

So let $\mathcal{A}$ be a category and Q a quiver. Our first goal is to consider representations of quivers in arbitrary categories. In what follows, we will omit the usual categorical definitions like additive, abelian and tensor categories, and functors between them; we refer to the definitions found in [7, 13].

Definition. *A representation of Q in $\mathcal{A}$ consists of*

- *a collection of objects $\{V_i\}, i \in Q_0$;*
- *a collection of morphisms $\{\varphi_a\}, a \in Q_1$, where $\varphi_a \in \mathrm{Hom}_{\mathcal{A}}(V_{t(a)}, V_{h(a)})$.*

The representations of Q in $\mathcal{A}$ is a category that we denote by $\mathrm{Rep}(Q, \mathcal{A})$.

It is interesting to observe that the category of representations $\mathrm{Rep}(Q, \mathcal{A})$ inherits some of the properties of the original category $\mathcal{A}$; in particular, one has the following result.

Theorem 1. *If $\mathcal{A}$ is additive (abelian) then* $\mathrm{Rep}(Q, \mathcal{A})$ *is also additive (abelian).*

Proof. We prove that if $\mathcal{A}$ is additive, then $\mathrm{Rep}(Q,\mathcal{A})$ is additive. To do that we have to check that $\mathrm{Rep}(Q,\mathcal{A})$ satisfies some conditions, see [7].

(i) $\mathrm{Rep}(Q,\mathcal{A})$ has a zero object.
Since $\mathcal{A}$ is additive, we can take the object $V_a = 0 \in \mathrm{Obj}(\mathcal{A}), \forall a \in Q_1$, and $\varphi_a = 0 \in \mathrm{Hom}_{\mathcal{A}}(0,0)$.

(ii) If $X,Y,Z \in \mathrm{Obj}(\mathcal{R})$ then $\mathrm{Hom}_{\mathcal{R}}(X,Y)$ is an abelian group and

$$\mathrm{Hom}_{\mathcal{R}}(X,Y) \times \mathrm{Hom}_{\mathcal{R}}(Y,Z) \to \mathrm{Hom}_{\mathcal{R}}(X,Z)$$

is bi-additive.
Let $X = (V,\phi)$, $Y = (W,\psi)$ and $Z = (U,\eta)$ be objects of $\mathcal{R} = \mathrm{Rep}(Q,\mathcal{A})$. As $\mathrm{Hom}_{\mathcal{A}}(V_i,W_i)$ is an abelian group for all $i \in Q_0$, $\mathrm{Hom}_{\mathcal{R}}(X,Y)$ is also an abelian group. It follows that

$$\mathrm{Hom}_{\mathcal{R}}(X,Y) \times \mathrm{Hom}_{\mathcal{R}}(Y,Z) \to \mathrm{Hom}_{\mathcal{R}}(X,Z)$$

is bi-additive because for each $i \in Q_0$, the pairing

$$\mathrm{Hom}_{\mathcal{A}}(V_i,W_i) \times \mathrm{Hom}_{\mathcal{A}}(W_i,U_i) \to \mathrm{Hom}_{\mathcal{A}}(V_i,U_i)$$

is bi-additive.

(iii) $\mathrm{Rep}(Q,\mathcal{A})$ has coproducts.
Let (V,ϕ) and (W,ψ) be objects of $\mathrm{Rep}(Q,\mathcal{A})$. As $\mathcal{A}$ is additive, for all $i \in Q_0$, $U_i = V_i \oplus W_i$ is an object of $\mathcal{A}$ with the maps $l_i : V_i \to U_i$ and $h_i : W_i \to U_i$.
We want to define the direct sum of (V,ϕ) and (W,ψ). Consider the diagram

$$\begin{tikzcd}
V_{t(a)} \arrow[r, "\phi_a"] \arrow[d, "l_{t(a)}"'] \arrow[dr, dashed, "c_a"] & V_{h(a)} \arrow[d, "l_{h(a)}"] \\
U_{t(a)} \arrow[r, dashed] & U_{h(a)} \\
W_{t(a)} \arrow[r, "\psi_a"'] \arrow[u, "h_{t(a)}"] \arrow[ur, dashed, "d_a"'] & W_{h(a)} \arrow[u, "h_{h(a)}"']
\end{tikzcd}$$

where $c_a = l_{h(a)} \circ \phi_a$ and $d_a = h_{h(a)} \circ \psi_a$.
By a property of coproducts, there exists an unique morphism

$$\eta_a : U_{t(a)} \to U_{h(a)}$$

such that $c_a = \eta_a l_{t(a)}$ and $d_a = \eta_a h_{t(a)}$. Then,

$$l_{h(a)}\phi_a = \eta_a l_{t(a)} \text{ and } h_{h(a)}\psi_a = \eta_a h_{t(a)}.$$

Let (U, η) be the representation $U_i = V_i \oplus W_i$ for $i \in Q_0$ and for each $a \in Q_1$, $\eta_a : U_{t(a)} \to U_{h(a)}$ as above. Then (U, η) is the direct sum of (V, ϕ) and (W, ψ) with the morphisms

$$L = \{l_i\}_{i \in Q_0} : (V, \phi) \to (U, \eta) \quad \text{and} \quad H = \{h_i\}_{i \in Q_0} : (W, \psi) \to (U, \eta).$$

Therefore $\mathrm{Rep}(Q, \mathcal{A})$ is an additive category.

If $\mathcal{A}$ is abelian, we have to verify other four conditions.

(1) Every morphism has kernel and cokernel.

Let $f : (V, \phi) \to (W, \psi)$ be a morphism. We want to define $\ker f$. As $\mathcal{A}$ is abelian, for each $i \in Q_0$, $f_i : V_i \to W_i$ is a morphism then it has a kernel (V_i', μ_i)

$$\mu_i : V_i' \to V_i, \text{ with } f_i \mu_i = 0, \forall i \in Q_0.$$

For each $a \in Q_1$ we have $f_{h(a)}(\phi_a \mu_{t(a)}) = 0$ then there is an unique morphism $\phi_a' : V_{t(a)}' \to V_{h(a)}'$ such that $\phi_a \mu_{t(a)} = \mu_{h(a)} \phi_a'$.

If we set

$$\mu = \{\mu_i\}_{i \in Q_0} : (V', \phi') \to (V, \phi)$$

where $V' = \{V_i'\}_{i \in Q_0}$ and $\phi' = \{\phi_a'\}_{a \in Q_1}$, the diagram commutes

$$\begin{tikzcd}
V_{t(a)}' \arrow[r, "\phi_a'"] \arrow[d, "\mu_{t(a)}"'] & V_{h(a)}' \arrow[d, "\mu_{h(a)}"] \\
V_{t(a)} \arrow[r, "\phi_a"] \arrow[d, "f_{t(a)}"'] & V_{h(a)} \arrow[d, "f_{h(a)}"] \\
W_{t(a)} \arrow[r, "\psi_a"'] & W_{h(a)}
\end{tikzcd}$$

then $f \circ \mu = 0$. One can see that $\ker f = ((V', \phi'), \mu)$ is in fact kernel of f.

Similarly one can see that f has cokernel.

(2) Every monomorphism is the kernel of its cokernel.

Let $f : (V, \phi) \to (W, \psi)$ be a monomorphism. Then $f_i : V_i \to W_i$ is a monomorphism for each $i \in Q_0$ and if $(W'_i, c_i) = \operatorname{coker} f_i$ we have $(V_i, f_i) = \ker c_i$, $\forall i \in Q_0$. As $c_{h(a)}\psi_a : W_{t(a)} \to W'_{h(a)}$ is such that $(c_{h(a)}\psi_a)f_{t(a)} = 0$ there is an unique $\psi'_a : W'_{t(a)} \to W'_{h(a)}$ with $c_{h(a)}\psi_a = c_{t(a)}\psi'_a$

$$\begin{tikzcd}
V_{t(a)} \arrow[r, "\phi_a"] \arrow[d, "f_{t(a)}"'] & V_{h(a)} \arrow[d, "f_{h(a)}"] \\
W_{t(a)} \arrow[r, "\psi_a"] \arrow[d, "c_{t(a)}"'] & W_{h(a)} \arrow[d, "c_{h(a)}"] \\
W'_{t(a)} \arrow[r, "\psi'_a"'] & W'_{h(a)}
\end{tikzcd}$$

and then $((W', \psi'), c) = \operatorname{coker} f$. We have $(V_i, f_i) = \ker c_i$ and therefore $((V, \phi), f) = \ker c$.

(3) Every epimorphism is cokernel of its kernel.

Let $f : (V, \phi) \to (W, \psi)$ be an epimorphism. Then for each $i \in Q_0$ $f_i : V_i \to W_i$ is an epimorphism. As $\mathcal{A}$ is abelian if $(V'_i, k_i) = \ker f_i$ we have $f_i k_i = 0, \forall i \in Q_0$. As $\phi_a k_{t(a)} : V'_{t(a)} \to V_{h(a)}$ is such that $f_{h(a)}(\phi_a k_{t(a)}) = 0$ there is an unique $\phi'_a : V'_{t(a)} \to V'_{h(a)}$, $a \in Q_1$, such that $\phi_a k_{t(a)} = k_{h(a)}\phi'_a$ and $((V', \phi'), k) = \ker f$.

$$\begin{tikzcd}
V'_{t(a)} \arrow[r, "\phi'_a"] \arrow[d, "k_{t(a)}"'] & V'_{h(a)} \arrow[d, "k_{h(a)}"] \\
V_{t(a)} \arrow[r, "\phi_a"] \arrow[d, "f_{t(a)}"'] & V_{h(a)} \arrow[d, "f_{h(a)}"] \\
W_{t(a)} \arrow[r, "\psi_a"'] & W_{h(a)}
\end{tikzcd}$$

We also have $(W_i, f_i) = \operatorname{coker} k_i$ and f_i is an epimorphism therefore $((W, \psi), f) = \operatorname{coker} k$ and follows the result.

(4) Every morphism can be written as composition of epimorphism and monomorphism.

Let $f : (V, \phi) \to (W, \psi)$ be a morphism. Since $\mathcal{A}$ is abelian we know that for each $i \in Q_0$, $f_i : V_i \to W_i$ can be written as $f_i = h_i g_i$, where $g_i : V_i \to W_i'$ is epimorphism and $h_i : W_i' \to W_i$ is monomorphism, with $W_i' \in \mathrm{Obj}(\mathcal{A})$. By a lemma the diagram

$$\begin{tikzcd} V_{t(a)} \arrow[r, "\phi_a"] \arrow[d, "f_{t(a)}"'] & V_{h(a)} \arrow[d, "f_{h(a)}"] \\ W_{t(a)} \arrow[r, "\psi_a"'] & W_{h(a)} \end{tikzcd}$$

commutes, then for each $a \in Q_1$ there is an unique

$$\psi_a' : W_{t(a)}' \to W_{h(a)}'$$

such that

$$\begin{tikzcd} V_{t(a)} \arrow[r, "\phi_a"] \arrow[d, "g_{t(a)}"'] & V_{h(a)} \arrow[d, "g_{h(a)}"] \\ W_{t(a)}' \arrow[r, "\psi_a'"] \arrow[d, "h_{t(a)}"'] & W_{h(a)}' \arrow[d, "h_{h(a)}"] \\ W_{t(a)} \arrow[r, "\psi_a"'] & W_{h(a)} \end{tikzcd}$$

commutes, that is, $\psi_a' g_{t(a)} = g_{h(a)} \phi_a$ and $\psi_a h_{t(a)} = h_{h(a)} \psi_a'$. Then if we take the representation (W', ϕ') and the morphisms $h = \{h_i\}_{i \in Q_0} : (W', \psi') \to (W, \psi)$ and $g : \{g_i\}_{i \in Q_0} (V, \phi) \to (W, \psi)$ follows that g is epimorphism, h is monomorphism and $f = h \circ g$.

$\square$

Remark 2. *We believe that other standard categorical properties of $\mathcal{A}$, like the existence of sufficiently many projectives and injectives, will also be inherited by the category of representations* $\mathrm{Rep}(Q, \mathcal{A})$.

If $\mathcal{A}$ is an additive category, one can also consider representations of quivers with relations in $\mathcal{A}$.

Definition. *Let $\mathcal{A}$ be an additive category, Q a quiver and $R = \{R_1, \ldots, R_m\}$ be a set of relations in Q, where $R_i = \sum_{j=1}^{n_i} p_j^i$, $p_j^i = a_{j1}^i \cdots a_{jl_j}^i$. A representation (V, ϕ) in $\mathcal{A}$ satisfies the relations if $\sum_{j=1}^{n_i} \phi_{p_j^i} = 0$ for each $i = 1, \cdots, m$, where $\phi_{p_j^i} = \phi_{a_{j1}^i} \cdots \phi_{a_{jl_j}^i}$.*

We denote by $\mathrm{Rep}((Q, R), \mathcal{A})$ the category of representations of the quiver with relations. It is not difficult to see that $\mathrm{Rep}((Q, R), \mathcal{A})$ is a full additive subcategory of $\mathrm{Rep}(Q, \mathcal{A})$.

2.2. **Induced functors.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories, Q a quiver and $\mathbf{F} : \mathcal{A} \to \mathcal{B}$ a functor. We can define a functor $\mathbf{F}_Q : \mathrm{Rep}(Q, \mathcal{A}) \to \mathrm{Rep}(Q, \mathcal{B})$ in the following way:

- given an object $(V, \phi) \in \mathrm{Rep}(Q, \mathcal{A})$, $\mathbf{F}_Q((V, \phi)) = (W, \psi)$ where $W_i = \mathbf{F}(V_i) \in \mathrm{Obj}(\mathcal{B}), i \in Q_0$, and $\psi_a = \mathbf{F}(\phi_a) \in \mathrm{Hom}_{\mathcal{B}}(W_{t(a)}, W_{h(a)}), a \in Q_1$;
- given a morphism $f : (V, \phi) \to (W, \psi)$, $f = \{f_i\}, f_i \in \mathrm{Hom}_{\mathcal{A}}(V_i, W_i)$ we define $\mathbf{F}_Q(f) = g$, where $g_i = \mathbf{F}(f_i) \in \mathrm{Hom}_{\mathcal{B}}(\mathbf{F}(V_i), \mathbf{F}(W_i)), i \in Q_0$.

It turns out that the induced functor $\mathbf{F}_Q$ also inherits properties of the functor original $\mathbf{F}$, as in the following Proposition.

Proposition 3. *Let $\mathbf{F} : \mathcal{A} \to \mathcal{B}$ be a functor.*

- *If $\mathbf{F}$ is an equivalence of categories, then so is the induced functor $\mathbf{F}_Q$ for any quiver Q;*
- *If $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $\mathbf{F}$ is an exact functor, then so is the induced functor $\mathbf{F}_Q$ for any quiver Q.*

Proof. First, suppose $\mathbf{F}$ is an equivalence of categories. To show that $\mathbf{F}_Q$ is equivalence of categories we must see that $\mathbf{F}_Q$ is fully faithful and essentially surjective.

Let $X = (V, \phi)$ and $Y = (W, \psi)$ be objects of $\mathrm{Rep}(Q, \mathcal{A})$, and take $f, g \in \mathrm{Hom}(X, Y)$ such that $\mathbf{F}_Q(f) = \mathbf{F}_Q(g)$, with

$$\mathbf{F}_Q : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathbf{F}(X), \mathbf{F}(Y)).$$

Then $\mathbf{F}(f_i) = \mathbf{F}(g_i), \forall i \in Q_0$, $f_i, g_i \in \mathrm{Hom}(V_i, W_i)$. Since $\mathbf{F}$ is faithful, $f_i = g_i, \forall i \in Q_0$ then $f = g$ therefore $\mathbf{F}_Q$ is also faithful.

If $\overline{g} \in \mathrm{Hom}(F_Q(X), F_Q(Y))$ where $\overline{g} = \{\overline{g}_i\}_{i \in Q_0}$, $\overline{g}_i \in \mathrm{Hom}(F(V_i), F(W_i))$, since $\mathbf{F}$ is full, for each $i \in Q_0$ there is a morphism $g_i \in \mathrm{Hom}(V_i, W_i)$ such that $\mathbf{F}(g_i) = \overline{g_i}$. Thus $\mathbf{F}(g) = \overline{g}$ where $g = \{g_i\}_{i \in Q_0}$, hence $\mathbf{F}_Q$ is full. We have shown that if $\mathbf{F}$ is fully faithful, then so is $\mathbf{F}_Q$.

Let $Y = (W, \psi) \in \mathrm{Obj}(\mathrm{Rep}(Q, \mathcal{B}))$. Since $\mathbf{F}$ is essentially surjective, there is an isomorphism $\lambda_i : W_i \to \mathbf{F}(V_i), \forall i \in Q_0$, $V_i \in \mathrm{Obj}(\mathcal{A})$. For each $a \in Q_1$ consider the map $\phi'_a = \lambda_{h(a)} \psi_a \lambda^{-1}_{t(a)}$. Then $\phi'_a \in \mathrm{Hom}(\mathbf{F}(V_{t(a)}), \mathbf{F}(V_{h(a)}))$ and since $\mathbf{F}$ is full, there is a map $\phi_a \in \mathrm{Hom}(V_{t(a)}, V_{h(a)})$ such that $\mathbf{F}(\phi_a) = \phi'_a$. Let $X = (V, \phi)$ where $V = \{V_i\}_{i \in Q_0}$ and $\phi = \{\phi_a\}_{a \in Q_1}$. Then $X \in \mathrm{Obj}(\mathrm{Rep}(Q, \mathcal{A}))$ and $\mathbf{F}(X) \simeq Y$ with isomorphism λ and so $\mathbf{F}_Q$ is essentially surjective, therefore $\mathbf{F}_Q$ is equivalence of categories.

Now suppose that $\mathbf{F}$ is an exact functor and let $X = (V, \phi)$, $Y = (W, \psi)$, $Z = (U, \lambda)$ be objects of $\mathrm{Rep}(Q, \mathcal{A})$, $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, Z)$ such that

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is a short exact sequence. Then, for each $a \in Q_1$, the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
V_{t(a)} \arrow[r, "\phi_a"] \arrow[d, "f_{t(a)}"'] & V_{h(a)} \arrow[d, "f_{h(a)}"] \\
W_{t(a)} \arrow[r, "\psi_a"] \arrow[d, "g_{t(a)}"'] & W_{h(a)} \arrow[d, "g_{h(a)}"] \\
U_{t(a)} \arrow[r, "\lambda_a"] \arrow[d] & U_{h(a)} \arrow[d] \\
0 & 0
\end{tikzcd}$$

Since $\mathbf{F}$ is an exact functor, the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & \mathbf{F}(V_{t(a)}) \arrow[r, "\mathbf{F}(\phi_a)"] \arrow[d, "\mathbf{F}(f_{t(a)})"'] & \mathbf{F}(V_{h(a)}) \arrow[d, "\mathbf{F}(f_{h(a)})"] \\
 & \mathbf{F}(W_{t(a)}) \arrow[r, "\mathbf{F}(\psi_a)"] \arrow[d, "\mathbf{F}(g_{t(a)})"'] & \mathbf{F}(W_{h(a)}) \arrow[d, "\mathbf{F}(g_{h(a)})"] \\
 & \mathbf{F}(U_{t(a)}) \arrow[r, "\mathbf{F}(\lambda_a)"] \arrow[d] & \mathbf{F}(U_{h(a)}) \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

is also commutative, then

$$0 \longrightarrow \mathbf{F}_Q(X) \xrightarrow{\mathbf{F}_Q(f)} \mathbf{F}_Q(Y) \xrightarrow{\mathbf{F}_Q(g)} \mathbf{F}_Q(Z) \longrightarrow 0$$

is a short exact sequence and therefore $\mathbf{F}_Q$ is an exact functor. □

3. Twisted linear representations of quivers

The concept of twisted representations of quivers was first introduced by Gothen and King [8, p. 88], motivated by certain problems involving vector bundles with extra structures, like Higgs bundles and holomorphic triples. Here, we will define twisted representations in an arbitrary tensor category, and, for representations in the category of vector spaces over a field, we will relate the category of twisted representations of Q with the category of representations of a different quiver $\tilde{Q}$.

3.1. Twisted representations of quivers.

Definition. *Let Q be a quiver and $\mathcal{A}$ a tensor category. Fix a collection $M = \{M_a\}_{a\in Q_1}$ of objects of $\mathcal{A}$. A* right M-twisted representation of Q *consists of*

- *a collection of objects $\{V_i \mid i \in Q_0\}$;*
- *a collection of morphisms $\{\varphi_a : V_{t(a)} \to V_{h(a)} \otimes M_a \mid a \in Q_1\}$.*

We also denote the representation by (V, ϕ). Alternatively, one could also consider the morphisms $\{\varphi_a : M_a \otimes V_{t(a)} \to V_{h(a)} \mid a \in Q_1\}$, leading to the notion of a left $M-$twisted representation of Q.

A morphism between two twisted representations (V, ϕ) and (W, ψ) in $\mathcal{A}$, is a collection of morphisms $f_i : V_i \to W_i$, $i \in Q_0$, such that the following diagram commutes for each $a \in Q_1$.

$$\begin{tikzcd}
V_{t(a)} \arrow[r, "\phi_a"] \arrow[d, "f_{t(a)}"'] & V_{h(a)} \otimes M_a \arrow[d, "f_{h(a)} \otimes 1_{M_a}"] \\
W_{t(a)} \arrow[r, "\psi_a"'] & W_{h(a)} \otimes M_a
\end{tikzcd}$$

that is,

$$(f_{h(a)} \otimes 1_{M_a}) \circ \phi_a = \psi_a \circ f_{t(a)},\ \forall a \in Q_1.$$

We have then a new category of right $M-$twisted representations of Q, denoted by $\mathrm{Rep}_M(Q, \mathcal{A})$.

Suppose Q has relations $R = \{R_1, \cdots, R_m\}$, where $R_i = \sum_{j=1}^{n_i} p_j^i$, and $p_j^i = a_{j1}^i \cdots a_{jl_j}^i$. Let $M = \{M_a\}$, $a \in Q_1$, be a collection of objects of $\mathcal{A}$, $M_{p_j^i} = M_{a_{j1}^i} \otimes M_{a_{j2}^i} \otimes \ldots \otimes M_{a_{jl_j}^i}$, $j = 1, \ldots n_i, i = 1, \ldots, m$ and $\tilde{M} = \bigotimes_{a \in Q_1} M_a$. Given a twisted representation (V, ϕ) we have induced maps

$$\tilde{\phi}_{p_j^i} : V_{t(p_j^i)} \to V_{h(p_j^i)} \otimes M_{p_j^i}$$

where

$$\tilde{\phi}_{p_j^i} = (\phi_{a_{j1}^i} \otimes 1_{M_{a_{j2}^i}} \otimes \ldots \otimes 1_{M_{a_{jl_j}^i}}) \circ (\phi_{a_{j2}^i} \otimes 1_{M_{a_{j3}^i}} \otimes \ldots \otimes 1_{M_{a_{jl_j}^i}}) \circ \ldots$$
$$\ldots \circ (\phi_{a_{jl_j-1}^i} \otimes 1_{M_{a_{jl_j}^i}}) \circ \phi_{a_{jl_j}^i}.$$

If $f_{p_j^i} : M_{p_j^i} \to \tilde{M}$ is the inclusion map and $\phi_{p_j^i} = (1_{V_{h(p_j^i)}} \otimes f_{p_j^i}) \circ \tilde{\phi}_{p_j^i}$, we have

$$\phi_{p_j^i} : V_{t(p_j^i)} \to V_{h(p_j^i)} \otimes \tilde{M},$$

for $j = 1, \ldots, n_i$

A twisted representation (V, ϕ) satisfies the relations if $\phi_{p_1^i} + \ldots + \phi_{p_{n_i}^i} = 0$, for $i = 1, \cdots, m$. We have again a new category, the category of $M-$twisted representations of the quiver Q with relations.

3.2. **Twisted representations of quivers.** Let Q be a quiver, $\mathcal{A}$ be the category of finite dimensional vector spaces over a field k and let $M = \{M_a\}, a \in Q_1$, be a collection of objects of $\mathcal{A}$. The main result of this section is to relate, as we mentioned before, the categories of twisted k-linear representations of a quiver with a category of k-linear representations of another quiver.

Theorem 4. *The category of $M-$twisted k-linear representations of Q is equivalent to the category of k-linear representations of $\tilde{Q}$, where $\tilde{Q}$ is obtained from Q in the following way;*

- *the set of vertices is the same, that is, $Q_0 = \tilde{Q}_0$;*
- *for each arrow $a \in Q_1$, $\tilde{Q}_1$ possesses $m = \dim M_a$ arrows, $a_1, \cdots, a_m$, such that $\tilde{t}(a_j) = t(a)$ and $\tilde{h}(a_j) = h(a)$, where $\tilde{t}$,$\tilde{h}$ are the tail and head maps of $\tilde{Q}$.*

Proof. Suppose that Q is $Q_0 = (1, 2)$ and $Q_1 = (a)$ with $t(a) = 1$ and $h(a) = 2$

$$1 \xrightarrow{a} 2 .$$

Fix M_a a $k-$vector space, $\dim M_a = m$. We show that $\mathcal{C} = \mathrm{Rep}_M(Q)$ is equivalent to $\mathcal{D} = \mathrm{Rep}(\tilde{Q})$, where $\tilde{Q} = (\tilde{Q}_0, \tilde{Q}_1)$ with $\tilde{Q}_0 = Q_0$ and $\tilde{Q}_1 = (a_1, \ldots, a_m)$, $t(a_i) = t(a)$ and $h(a_i) = h(a)$, for $i = 1, \ldots, m$.

We construct a functor that is an equivalence of categories. Let $\{V, W\}$,
$\{\phi : V \to W \otimes M_a\}$ be a representation of Q. Since $L(V, W \otimes M_a) \simeq L(V, W) \otimes M_a$, if $\{h_1, \ldots, h_m\}$ is a basis for M_a and $\phi \in L(V, W \otimes M_a)$, there are $\phi_1, \cdots, \phi_m \in L(V, W)$ such that

$$\phi = \sum_{i=1}^{m} \phi_i \otimes h_i. \tag{1}$$

Fix an order in the set of arrows between two vertices of the quiver $\tilde{Q}$ and consider the functor

$$\mathbf{F} : \mathrm{Rep}_M(Q) \to \mathrm{Rep}(\tilde{Q})$$

such that

- for each object $X = (\{V_{t(a)}, V_{h(a)}\}, \phi) \in \mathrm{Rep}_M(Q)$

  $$\mathbf{F}(X) = \mathbf{F}(\{V_{t(a)}, V_{h(a)}\}, \phi) = (\{V_{t(a)}, V_{t(a)}\}, \tilde{\phi})$$

  where $\tilde{\phi} = \{\phi_1, \ldots, \phi_m\}$ are obtained as in (1) and ϕ_n is associated to the $n-$th arrow of $\tilde{Q}$;
- for each morphism $f = \{f_{t(a)}, f_{h(a)}\}$ between $X = (\{V_{t(a)}, V_{h(a)}\}, \phi)$ and $Y = (\{W_{t(a)}, W_{h(a)}\}, \psi)$

  $$\begin{tikzcd} V_{t(a)} \arrow[r, "\phi"] \arrow[d, "f_{t(a)}"'] & V_{h(a)} \otimes M_a \arrow[d, "f_{h(a)} \otimes 1_{M_a}"] \\ W_{t(a)} \arrow[r, "\psi"'] & W_{h(a)} \otimes M_a \end{tikzcd}$$

  we define $\mathbf{F}(f) = f$.

It's easy to see that $f = \{f_{t(a)}, f_{h(a)}\}$ is a morphism between the representations $(\{V_{t(a)}, V_{h(a)}\}, \{\phi_1, \ldots, \phi_m\})$ and $(\{W_{t(a)}, W_{h(a)}\}, \{\psi_1, \ldots \psi_m\})$, and that $\mathbf{F}$ is a functor. Now we must show that $\mathbf{F}$ is an equivalence of categories. To see this is sufficient to show that $\mathbf{F}$ is fully faithful functor and essentially surjective, see [7, p. 71].

(1) $\mathbf{F} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{F}(X), \mathbf{F}(Y))$ is injective and surjective. Clearly $\mathbf{F}$ is injective. Let $g \in \mathrm{Hom}_{\mathcal{D}}(\mathbf{F}(X), \mathbf{F}(Y))$, $g = \{g_i\}_{i \in Q_0}$ such that

$$\psi_i \circ g_{t(a)} = g_{h(a)} \circ \phi_i, \ i = 1, \ldots, m.$$

Then $f = \{g_i\}_{i \in Q_0} \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is such that $\mathbf{F}(f) = g$. In fact

$$\psi g_{t(a)} = (\sum_{j=1}^{m} \psi_j \otimes h_j) g_{t(a)} = \sum_{j=1}^{m} ((\psi_j \otimes h_j) g_{t(a)}) = \sum_{j=1}^{m} \psi_j g_{t(a)} \otimes h_j =$$

$$\sum_{j=1}^{m} g_{h(a)} \phi_j \otimes h_j = \sum_{j=1}^{m} g_{h(a)} \phi_j \otimes 1_{M_a} h_j = \sum_{j=1}^{m} (g_{h(a)} \otimes 1_{M_a})(\phi_j \otimes h_j) =$$

$$(g_{h(a)} \otimes 1_{M_a})(\sum_{j=1}^{m} \phi_j \otimes h_j) = (g_{h(a)} \otimes 1_{M_a})\phi.$$

Therefore $\mathbf{F}$ is fully faithful.

(2) Let $\overline{Y} \in \mathrm{Obj}(\mathcal{D})$, $\overline{Y} = (V, \overline{\phi})$ where $V = \{V_{t(a)}, V_{h(a)}\}$ and $\overline{\phi} = \{\phi_1, \ldots, \phi_m\}$ with $\phi_j : V_{t(a)} \to V_{h(a)}$, $j = 1, \ldots, m$. Let $X = (V, \phi)$ with

$$\phi = \sum_{j=1}^{m} \phi_j \otimes h_j.$$

Then $\overline{Y} = \mathbf{F}(X)$. Therefore $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories. Since the construction of the proof depends on the arrow, the proof for the general quiver follows from performing this construction for each arrow.

□

Remark 5. *The particular functor constructed in the proof of the theorem depend on two choices: the choice of an order in the set of arrows between two vertices of the quiver $\tilde{Q}$, and the choice of bases for the vector spaces M_a for each $a \in Q_1$. It is reasonable to expect that different choices lead to naturally isomorphic functors, although the authors have not been able to establish this claim.*

Proposition 6. *Let $M = \{M_a\}_{a \in Q_1}$ be a collection of finite dimensional $k-$vector spaces. Given $M' = \{M'_a\}_{a \in Q_1}$ a collection of vector subspaces, $M'_a \subset M_a$, $a \in Q_1$, there is a fully faithful functor $\mathbf{F} : \mathrm{Rep}_{M'}(Q) \to \mathrm{Rep}_M(Q)$.*

Proof. Let $\mathcal{C} = \mathrm{Rep}_{M'}(Q)$ and $\mathcal{D} = \mathrm{Rep}_M(Q)$. Let $X = (V, \phi) \in \mathrm{Obj}(\mathcal{C})$ then $\phi_a : V_{t(a)} \to V_{h(a)} \otimes M'_a$ for each $a \in Q_1$. Let $\varepsilon_a : M'_a \to M_a$ be the inclusion map. Consider the map

$$\overline{\phi}_a : V_{t(a)} \to V_{h(a)} \otimes M_a$$

given by $\overline{\phi}_a = (1_{V_{h(a)}} \otimes \varepsilon_a) \circ \phi_a$ for each $a \in Q_1$. We define the functor $\mathbf{F} : \mathcal{C} \to \mathcal{D}$

- for $X = (V, \phi) \in \mathrm{Obj}(\mathcal{C})$ we have

$$\mathbf{F}(X) = (V, \overline{\phi})$$

where $\overline{\phi}_a = (1_{V_{h(a)}} \otimes \varepsilon_a) \circ \phi_a, a \in Q_1$;
- if $f : X \to Y$ is a morphism and $X = (V, \phi)$, $Y = (W, \psi)$ are objects of $\mathcal{C}$ then $\mathbf{F}(f) = f$ is a morphism between $\mathbf{F}(V, \phi)$ and $\mathbf{F}(W, \psi)$.

In fact we must show that the diagram bellow is commutative

$$\begin{tikzcd}
V_{t(a)} \arrow[r, "\overline{\phi}_a"] \arrow[d, "f_{t(a)}"'] & V_{h(a)} \otimes M_a \arrow[d, "f_{h(a)} \otimes 1_{M_a}"] \\
W_{t(a)} \arrow[r, "\overline{\psi}_a"'] & W_{h(a)} \otimes M_a
\end{tikzcd}$$

We have

$$\begin{aligned}
\overline{\psi}_a f_{t(a)} &= ((1_{W_{h(a)}} \otimes \varepsilon_a)\psi_a) f_{t(a)} = (1_{W_{h(a)}} \otimes \varepsilon_a)(\psi_a f_{t(a)}) \\
&= (1_{W_{h(a)}} \otimes \varepsilon_a)(f_{h(a)} \otimes 1_{M'_a})\phi_a
\end{aligned}$$

and

$$(f_{h(a)} \otimes 1_{M_a})\overline{\phi}_a = (f_{h(a)} \otimes 1_{M_a})(1_{V_{h(a)}} \otimes \varepsilon_a)\phi_a.$$

Since $1_{W_{h(a)}} f_{h(a)} = f_{h(a)} 1_{V_{h(a)}}$ and $\varepsilon_a 1_{M'_a} = 1_{M_a}\varepsilon_a$ note that

$$(1_{W_{h(a)}} \otimes \varepsilon_a)(f_{h(a)} \otimes 1_{M'_a}) = (f_{h(a)} \otimes 1_{M_a})(1_{V_{h(a)}} \otimes \varepsilon_a)$$

then

$$\overline{\psi}_a f_{t(a)} = (f_{h(a)} \otimes 1_{M_a})\overline{\phi}_a$$

and $\mathbf{F}(f) \in \mathrm{Hom}_{\mathcal{D}}(\mathbf{F}(X), \mathbf{F}(Y))$.

If $g : Y \to Z$ is a morphism where $Z = (U, \eta)$ then is easy to see that $gf : X \to Z$ is well defined and that $\mathbf{F}(fg) = \mathbf{F}(f)\mathbf{F}(g)$ e therefore $\mathbf{F}$ is a functor.

Now we have to show that $\mathbf{F}$ is fully faithful, that is, for each object $X, Y \in \mathrm{Obj}(\mathcal{C})$

$$\mathbf{F} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{F}(X), \mathbf{F}(Y))$$

is surjective and injective.

Let $X = (V, \phi)$ and $Y = (W, \psi)$ be objects of $\mathrm{Rep}_{M'}(Q)$ and let $g = \{g_i\}_{i \in Q_0} \in \mathrm{Hom}_{\mathcal{D}}(\mathbf{F}(X), \mathbf{F}(Y))$. Then the diagram commutes

$$\begin{tikzcd} V_{t(a)} \arrow[r, "\overline{\phi}_a"] \arrow[d, "g_{t(a)}"'] & V_{h(a)} \otimes M_a \arrow[d, "g_{h(a)} \otimes 1_{M_a}"] \\ W_{t(a)} \arrow[r, "\overline{\psi}_a"'] & W_{h(a)} \otimes M_a \end{tikzcd}$$

that is,

$$\overline{\psi}_a g_{t(a)} = (g_{h(a)} \otimes 1_{M_a}) \overline{\phi}_a, \ \forall a \in Q_1,$$

then

$$((1_{W_{h(a)}} \otimes \varepsilon_a)\psi_a) g_{t(a)} = (g_{h(a)} \otimes 1_{M_a})((1_{V_{h(a)}} \otimes \varepsilon_a)\phi_a)$$

and so

$$(1_{W_{h(a)}} \otimes \varepsilon_a)(\psi_a g_{t(a)}) = (g_{h(a)} \otimes 1_{M_a})(1_{V_{h(a)}} \otimes \varepsilon_a)\phi_a.$$

Note that

$$(g_{h(a)} \otimes 1_{M_a})(1_{V_{h(a)}} \otimes \varepsilon_a) = (1_{W_{h(a)}} \otimes \varepsilon_a)(g_{h(a)} \otimes 1_{M'_a})$$

then

$$(1_{W_{h(a)}} \otimes \varepsilon_a)(\psi_a g_{t(a)}) = (1_{W_{h(a)}} \otimes \varepsilon_a)((g_{h(a)} \otimes 1_{M'_a})\phi_a).$$

We have that $f = \{g_i\}_{i \in Q_1} \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is such that $\mathbf{F}(f) = g$ then $\mathbf{F} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{F}(X), \mathbf{F}(Y))$ is surjective. Clearly $\mathbf{F}$ is injective therefore is fully faithful. $\square$

In the previous Proposition one may choose each M'_a to be a 1-dimensional subspace of M_a; together with Theorem 4, we have the following statement.

Corollary 7. $\mathrm{Rep}(Q)$ *is equivalent to a full subcategory of* $\mathrm{Rep}_M(Q)$.

4. Monads on projective varieties and representations of quivers

Representations of quivers arise naturally in certain important problems concerning vector bundles on projective varieties. In this Section, we will discuss two such examples: first, we observe that monads over a projective variety X can be regarded as representations of quivers in the category of vector bundles over X; we then specialize to linear monads and linear bundles, and show that these form a category which is equivalent to a certain subcategory of a category of twisted linear representations.

We must first recall the following definition of an exact subcategory of an abelian category.

Definition. *Let $\mathcal{A}$ be an abelian category. A full, additive subcategory $\mathcal{E}$ of $\mathcal{A}$ is said to be exact if the following two conditions hold:*

(1) *$\mathcal{E}$ is closed under extensions, i.e. if Y and Z are objects of $\mathcal{E}$, then any extension of Z by Y is also objects of $\mathcal{E}$;*
(2) *$\mathcal{E}$ is closed under direct summands, i.e. if X is an object of $\mathcal{E}$ and $X \simeq Y \oplus Z$, then Y and Z are also objects of $\mathcal{E}$.*

In what follows, we will be particularly interested in the following quiver with relation

$$Q = \left\{ \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \right\} \;\; R = \{ba\} \ ,$$

which we denote (A_3, ba). If X is a *projective variety*, i.e. a projective scheme over an algebraically closed field $\mathbb{F}$ together with a given very ample invertible sheaf denoted by $\mathcal{O}_X(1)$, we set $\mathcal{M}(X) := \mathrm{Rep}((A_3, ba), \mathrm{Coh}(X))$, where $\mathrm{Coh}(X)$ is the category of coherent sheaves of $\mathcal{O}_X$-modules on X. Let $\mathcal{V}(X)$ be the category of locally-free sheaves on X.

Furthermore, For any coherent sheaf E on X, we set $E(k) := E \otimes \mathcal{O}_X(k)$ and $H^p_*(E) := \oplus_{k\in\mathbb{Z}} H^p(E(k))$. Finally, ω_X denotes the dualizing sheaf on X.

4.1. **Monads on projective varieties.** We start by recalling the notion of a Horrocks monad, cf. [1, Definition 2.2] and [14, page 239].

Definition. *A* monad *on X is a complex of locally free sheaves*

$$\mathrm{M}_\bullet \ : \ M_0 \xrightarrow{\alpha} M_1 \xrightarrow{\beta} M_2$$

such that β is surjective, α is injective. A monad is said to be Horrocks *if in addition*

(i) $M_0 = \oplus_{i=1}^{r}\ \omega_X(k_i)$ *for* $k_i \in \mathbb{Z}$*;*
(ii) $M_2 = \oplus_{j=1}^{s}\ \mathcal{O}_X(l_j)$ *for* $l_j \in \mathbb{Z}$*;*
(iii) $H^1_*(M_1) = H^{n-1}_*(M_1) = 0$.

Remark 8. *Notice that the above definition is weaker than* [12, Definition 2.1]*, where the maps β and α are assumed to be locally right-invertible and locally left-invertible, respectively. In this case, the cohomology sheaf $E = \ker\beta/\operatorname{im}\alpha$ is only a coherent sheaf (not necessarily locally-free).*

A morphism between two monads is simply a morphism of complexes. With these definitions, note that Horrocks monads on a projective scheme X form a category, denoted $\mathcal{H}(X)$. It is easy to see that $\mathcal{H}(X)$ is a full, additive subcategory of $\mathcal{M}(X)$, but more is true when one restricts the class of schemes under consideration.

Definition. *A projective variety $X \hookrightarrow \mathbb{P}^n$ of pure dimension d is* arithmetically Cohen-Macaulay *(ACM) if its homogeneous coordinate ring $S(X)$ is a Cohen-Macaulay ring.*

This is equivalent to saying that $H^1_*(\mathbb{P}^n, \mathcal{I}_X) = 0$ (where $\mathcal{I}_X$ is the saturated ideal of X) and $H^p_*(\mathcal{O}_X) = 0$ for every $1 \leq p \leq d-1$ [5]. In particular, $S(X) = H^0_*(\mathcal{O}_X)$. For instance, every complete intersection scheme $X \subset \mathbb{P}^n$ is ACM. Note that if X is ACM, then $H^p_*(\omega_X) = 0$ for $1 \leq p \leq d-1$, by Serre duality. Note also that $H^p_*(E)$ is a graded $S(X)$-module.

We then have the following proposition.

Proposition 9. *The category $\mathcal{H}(X)$ of Horrocks' monads on a non-singular ACM projective scheme X is an exact subcategory of $\mathcal{M}(X)$.*

Proof. Consider the following two objects of $\mathcal{M}(X)$

$$\mathrm{M}_\bullet\ :\ M_0 \xrightarrow{\alpha_M} M_1 \xrightarrow{\beta_M} M_2 \text{ and} \mathrm{N}_\bullet\ :\ N_0 \xrightarrow{\alpha_N} N_1 \xrightarrow{\beta_N} N_2$$

The direct sum $\mathrm{M}_\bullet \oplus \mathrm{N}_\bullet$ is the complex

$$M_0 \oplus N_0 \xrightarrow{\alpha_M \oplus \alpha_N} M_1 \oplus N_1 \xrightarrow{\beta_M \oplus \beta_N} M_2 \oplus N_2\ .$$

It is easy to see from the definition that if $\mathrm{M}_\bullet \oplus \mathrm{N}_\bullet$ is Horrocks, then $\mathrm{M}_\bullet$ and $\mathrm{N}_\bullet$ must also be Horrocks; in other words, $\mathcal{H}(X)$ is closed under direct summands.

To see that $\mathcal{H}(X)$ is closed under extensions, assume that $\mathrm{M}_\bullet$ and $\mathrm{N}_\bullet$ as above are two Horrocks monads. An extention of $\mathrm{M}_\bullet$ by $\mathrm{N}_\bullet$ is an object

$$\mathrm{L}_\bullet \ : \ L_0 \xrightarrow{\alpha_L} L_1 \xrightarrow{\beta_L} L_2$$

of $\mathcal{M}(X)$ such that the following diagram is commutative

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ N_0 \arrow[r, "\alpha_N"] \arrow[d, "g_1"] & N_1 \arrow[r, "\beta_N"] \arrow[d, "h_1"] & N_2 \arrow[d, "l_1"] \\ L_0 \arrow[r, "\alpha_L"] \arrow[d, "g_2"] & L_1 \arrow[r, "\beta_L"] \arrow[d, "h_2"] & L_2 \arrow[d, "l_2"] \\ M_0 \arrow[r, "\alpha_M"] \arrow[d] & M_1 \arrow[r, "\beta_M"] \arrow[d] & M_2 \arrow[d] \\ 0 & 0 & 0 \end{tikzcd} \tag{2}$$

in which the columns are exact. Chasing diagrams, one easily shows that α_L is injective, and β_L is surjective.

Notice also that L_k is an extension of M_k by N_k. Since X is an ACM scheme, it follows that the first and last columns split as exact sequences, giving maps $\tilde{g_2} : A_2 \to A$ and $\tilde{l_2} : C_2 \to C$ such that . Moreover, $L_0 = M_0 \oplus N_0$ and $L_2 = M_2 \oplus N_2$, hence L_0 and L_2 are also sums of line bundles of the form $\mathcal{O}_X(l_j)$ and $\omega_X(k_i)$, respectively. Looking at the cohomology sequence associated to the middle column, it is easy to see that $H^1_*(M_1) = H^1_*(N_1) = 0$ forces $H^1_*(L_1) = 0$, while $H^{n-1}_*(M_1) = H^{n-1}_*(N_1) = 0$ forces $H^{n-1}_*(L_1) = 0$.

We check that $\beta_L\alpha_L = 0$; first, notice that any local section σ of L_0 can be written as a sum $g_1(\sigma_1) + \tilde{g_2}(\sigma_2)$ with $\sigma_1 \in N_0$ and $\sigma_2 \in M_0$. Then

$$\beta_L\alpha_L(\sigma) = \beta_L\alpha_L g_1(\sigma_1) + \beta_L\alpha_L\tilde{g_2}(\sigma_2) = l_1\beta_N\alpha_N(\sigma_1) + \tilde{l_2}\beta_M\alpha_M(\sigma_2) = 0$$

since $\beta_N\alpha_N = \beta_M\alpha_M = 0$. So the middle row is a Horrocks' monad, as desired. □

Additionally, one can show that if X be a nonsingular ACM projective scheme of dimension $n \geq 3$ and, then the functor that associates each Horrocks' monad to its cohomology sheaf $E = \ker\beta/\mathrm{im}\ \alpha$ is additive, exact and full, cf. [12, Theorem 2.6].

4.2. **Linear bundles on projective spaces.** The following definition is motivated by [10], and it generalizes the concept of mathematical instanton bundle on $\mathbb{P}^{2m+1}$ introduced by Okonek and Spindler in [15, Definition 1.1].

Definition. *A torsion-free coherent sheaf E on $\mathbb{P}^n$ ($n \geq 2$) is called linear if it satisfies the following cohomological conditions:*

(1) *for $n \geq 2$, $H^0(E(-1)) = H^n(E(-n)) = 0$;*
(2) *for $n \geq 3$, $H^1(E(-2)) = H^{n-1}(E(1-n)) = 0$;*
(3) *for $n \geq 4$, $H^p(E(k)) = 0$, $2 \leq p \leq n-2$ and $\forall k$.*

Linear sheaves such that $c_1(E) = 0$ are called instanton sheaves.

Proposition 10. *Linear sheaves form an exact subcategory $\mathcal{L}(\mathbb{P}^n)$ of Coh($\mathbb{P}^n$).*

Proof. It is easy to see from the Definition that if E_1 and E_2 are linear sheaves, then any sheaf F in the exact sequence

$$0 \to E_1 \to F \to E_2 \to 0$$

is also linear, so $\mathcal{L}(\mathbb{P}^n)$ is closed under extensions. Assuming that $F = E_1 \oplus E_2$, we have that $H^p(F(k)) = H^p(E_1(k)) \oplus H^p(E_2(k))$, so closure under direct summands follows easily. □

Remark 11. *Notice that instanton sheaves form an additive subcategory of Coh($\mathbb{P}^n$) which is closed under extensions, but not closed under direct summands.*

Several properties of linear torsion-free sheaves on projective spaces are discussed in [10]; see also [11, 12] for properties of linear bundles over more general algebraic varieties.

Recall that a monad on $\mathbb{P}^n$ of the form

$$0 \to V_1 \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} V_2 \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\beta} V_3 \otimes \mathcal{O}_{\mathbb{P}^n}(1) \to 0\ , \qquad (3)$$

where V_k are vector spaces, is called a linear monad. Moreover, the cohomology of a linear monad is a torsion-free sheaf if and only if there is a closed subvariety $\Sigma \subset \mathbb{P}^n$ of codimension at least two such that the localized map $\alpha(x) : V_1 \to V_2$ is injective for each point $x \in (\mathbb{P}^n \setminus \Sigma)$, see [10, Proposition 4].

The most relevant fact is the following key result relating linear sheaves and linear monads, cf. Proposition 2 and Theorem 3 in [10].

Theorem 12. *If E is a linear torsion-free sheaf on $\mathbb{P}^n$, then E is isomorphic to the cohomology of the linear monad:*

$$\begin{aligned} 0 \;\to\;& H^1(E \otimes \Omega^2_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \to \\ \to\;& H^1(E \otimes \Omega^1_{\mathbb{P}^n}) \otimes \mathcal{O}_{\mathbb{P}^n} \to H^1(E(-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(1) \to 0\ . \end{aligned} \tag{4}$$

Conversely, the cohomology of a linear monad is a linear sheaf.

We will also need the following result, which is a special case of [12, Theorem 2.5]; see also [14, Lemma II.4.1.3].

Lemma 13. *If E and F are the cohomology sheaves of two linear monads $\mathcal{V}_\bullet(E)$ and $\mathcal{V}_\bullet(F)$, respectively, then the map that associates to each homomorphism of monads the corresponding homomorphism of cohomology sheaves is bijective.*

Now let us turn our attention to the relation between linear sheaves and twisted representations of the quiver (A_3, ba). Indeed, let $M = H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ and note that the maps α and β above can be regarded as matrices of linear polynomials, i.e. $\alpha \in \mathrm{Hom}(V_1, V_2) \otimes M$ and $\beta \in \mathrm{Hom}(V_2, V_3) \otimes M$. Therefore linear monads are in 1-1 correspondence with twisted representations of the quiver (A_3, ba)

$$V_1 \xrightarrow[\alpha]{M} V_2 \xrightarrow[\beta]{M} V_3$$

for which the map α is injective and β is surjective as maps of sheaves. Such representation will be called *admissible*; it is easy to see that admissible representations for a full additive subcategory, denoted denoted $\mathcal{A}(n)$, of $\mathrm{Rep}_M(A_3, ba)$, the category of all M-twisted representations of (A_3, ba). It is also worth noticing that the category $\mathrm{Rep}_M(A_3, ba)$ is equivalent to the category of linear representations of a different quiver with relations, providing a version of Theorem 4 for the quiver with relation (A_3, ba). More precisely, consider the following quiver, which we denote by Γ_n:

$$\bullet \underset{n+1}{\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}} \bullet \underset{n+1}{\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}} \bullet \tag{5}$$

with $\dim M = n{+}1$ arrows between each vertex. The $n{+}1$ arrows from the first to the second vertices are denoted $a_0, \ldots, a_n$, while the $n+1$ arrows from the second to the third vertices are denoted $b_0, \ldots, b_n$; we impose the following relations:

$$R_{k,l} \ : \ a_k b_l + a_l b_k = 0 \ , \ k, l = 0, \ldots, n \ . \tag{6}$$

Let $\mathbf{R}$ be the set of all relations $R_{k,l}$, $k \leq l$.

Proposition 14. *The equivalence functor* $\mathbf{F} : \mathrm{Rep}_M(A_3) \to \mathrm{Rep}(\Gamma_n)$ *constructed in the proof of Theorem 4 induces an equivalence of categories* $\mathbf{F} : \mathrm{Rep}_M(A_3, ba) \to \mathrm{Rep}(\Gamma_n, \mathbf{R})$*. Moreover, if* $V = \{(V_1, V_2, V_3), (\alpha, \beta)\}$ *is an admissible M-twisted representation of* (A_3, ba)*, then* $\mathbf{F}(V) = \{(V_1, V_2, V_3), (a_0, \ldots, a_n, b_0, \ldots, b_n)\}$ *is such that*

(1) *there exists* $(\mu_0, \ldots, \mu_n) \in \mathbb{C}^{n+1}$ *such that the map* $\mu_0 a_0 + \cdots + \mu_n a_n$ *is injective;*
(2) *the map* $\lambda_0 b_0 + \cdots + \lambda_n b_n$ *is surjective for every* $(\lambda_0, \ldots, \lambda_n) \in \mathbb{C}^{n+1}$.

Proof. Choose homogeneous coordinates $[x_0 : \cdots : x_n]$ in $\mathbb{P}^n$; that induces a natural of basis for $M = H^0(\mathcal{O}_{\mathbb{P}^n}(1))$, with respect to which the sheaf maps α and β can be written as follows:

$$\alpha = a_0 x_0 + \cdots + a_n x_n \text{ and } \beta = b_0 x_0 + \cdots + b_n x_n,$$

A straightforward calculation shows that $\beta\alpha = 0$ if and only if $a_k b_l + b_l a_k = 0$ for each $k, l = 0, \ldots, n$, matching the relations in equation (6). This concludes the proof of the first statement.

Finally, with the above notation in mind, the two conditions in the second part of the Proposition are easily seen to be equivalent to the injectivity of α and the surjectivity of β. □

We are finally in position to prove the main result of this Section.

Theorem 15. $\mathcal{A}(n)$ *is an exact subcategory of* $\mathrm{Rep}_M(A_3, ba)$*, which is equivalent to* $\mathcal{L}(\mathbb{P}^n)$*.*

Proof. The first statement is clear from the definitions. For the second statement, we define a functor:

$$\mathbf{G} : \mathcal{A}(\Gamma_n) \longrightarrow \mathcal{L}(\mathbb{P}^n)$$

as follows. Given a M-twisted representation $V = \{(V_1, V_2, V_3), (\alpha, \beta)\}$ of (A_3, ba), we form the complex of sheaves:

$$V_1 \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} V_2 \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\beta} V_3 \otimes \mathcal{O}_{\mathbb{P}^n}(1).$$

As we mentioned above, V is admissible if and only if the above complex is a linear monad with torsion-free cohomology sheaf (see [10, Proposition 4]), and we define $\mathbf{G}(V)$ as its cohomology sheaf. A morphism between admissible representations will induce a morphism between the corresponding linear monads, which in turn induces a morphism between the corresponding linear sheaves.

Now Theorem 12 implies that the functor $\mathbf{G}$ is essentially surjective, while Lemma 13 implies that $\mathbf{G}$ is fully faithful. This completes the proof. □

Theorem 15 allows us to translate geometric properties of sheaves into algebraic properties of the corresponding quiver representations, and vice versa. For example, note that the simple representation of $(\Gamma_n, \mathbf{R})$ with dimension vector $(0, 1, 0)$ corresponds, via the above functor, to the trivial line bundle $\mathcal{O}_{\mathbb{P}^n}$. The two other simple representations of Γ_n are not admissible.

Furthermore, given a representation R in $\mathcal{A}(\Gamma_n)$ of dimension vector (v_1, v_2, v_3), the Chern character of the linear sheaf $\mathbf{F}(R)$ is given by:

$$\mathrm{ch}(\mathbf{G}(R)) = v_2 - v_3 \cdot \mathrm{ch}(\mathcal{O}_{\mathbb{P}^n}(1)) - v_1 \cdot \mathrm{ch}(\mathcal{O}_{\mathbb{P}^n}(-1)) \ ;$$

in particular

$$\mathrm{rk}(\mathbf{G}(R)) = v_2 - v_3 - v_1 \text{ and } c_1(\mathbf{G}(R)) = v_1 - v_3 \ .$$

Therefore, rank r instanton sheaves correspond to representations with dimension vectors of the form $(c, r + 2c, c)$. The integer c is called the charge, of the corresponding instanton sheaf.

It follows from [6, Main Theorem] and the Theorem above that there exists an admissible representation of dimension vector (v_1, v_2, v_2) if and only if at least one of the following two conditions hold:

- $v_2 \geq 2v_3 + n - 1$ and $v_2 \geq v_1 + v_3$;
- $v_2 \geq v_1 + v_3 + n$.

Finally, the following interesting statement is an easy consequence of [10, Theorem 22] and Theorem 15 above.

Lemma 16. *Every admissible representation R of Γ_n with dimension vector $(c, n-1+2c, c)$ where $c \geq 1$ is Schurian, i.e.* $\mathrm{Hom}(R,R) = \mathbb{C}$.

As a next step, it would be interesting to study the possible dimension vectors of indecomposable M-twisted representations of (A_3, ba) (à la Kac's theorem), and in this way find the possible rank and charge of indecomposable instanton sheaves. We also expect to be able to establish new properties of the moduli spaces of linear sheaves by considering the moduli spaces of the corresponding quivers representations.

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